

## Letters and Comments

# Entropy change for the irreversible heat transfer between two finite objects

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## Abstract

A positive entropy change is verified for an isolated system of two blocks of different initial temperatures and of different but finite heat capacities that are brought into contact with each other and allowed to fully thermalize.

**Keywords:** thermalization, entropy, heat capacity, second law of thermodynamics

A recent article by Lima considers the timescale on which an initially hot object thermalizes with an initially cold object when they are brought into contact with each other inside an insulated box [1]. For simplicity, both objects are assumed to have temperature-independent heat capacities. Another interesting aspect of this problem is to compute the entropy change during the process. Textbooks typically only cover the cases where either the objects have identical heat capacity, or one of the objects is a thermal reservoir (i.e., having infinite heat capacity) so that its temperature remains constant [2]. It is a useful exercise to algebraically verify the second law of thermodynamics for the more general case where the two objects have unequal heat capacities, both of which are finite. (From a calculus point of view, each time an increment of heat  $dQ$  is transferred from object A at variable temperature  $T_A$  to object B at temperature  $T_B$ , the entropy change of the Universe is  $dS = -dQ/T_A + dQ/T_B$  which is positive whenever  $T_A > T_B$ .)

As sketched in figure 1 of Lima [1], the two blocks have initial absolute temperatures (in kelvin) of  $T_1$  and  $T_2$  where  $T_1 > T_2 > 0$ . Denote their respective heat capacities as  $C_1$  and  $C_2$ . If the common final equilibrium temperature of the two blocks is  $T_{eq}$ , then the total amount of heat transferred from block 1 to 2 is

$$Q = C_1(T_1 - T_{eq}) = C_2(T_{eq} - T_2) \quad (1)$$

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which can be rearranged to obtain

$$T_{\text{eq}} = \frac{C_1 T_1 + C_2 T_2}{C_1 + C_2}. \quad (2)$$

Block 1 loses entropy while block 2 gains entropy, so that the net entropy change of this isolated system is [2]

$$\Delta S = C_2 \ln \frac{T_{\text{eq}}}{T_2} - C_1 \ln \frac{T_1}{T_{\text{eq}}}. \quad (3)$$

The challenge now becomes to show that  $\Delta S > 0$  for *any* positive values of  $C_1$ ,  $C_2$ ,  $T_1$ , and  $T_2$  provided only that  $T_1 > T_2$ . Substituting equation (2) into (3), the goal is to verify the inequality

$$C_2 \ln \frac{C_1 T_1 + C_2 T_2}{C_1 T_2 + C_2 T_2} \stackrel{?}{>} C_1 \ln \frac{C_1 T_1 + C_2 T_1}{C_1 T_1 + C_2 T_2}. \quad (4)$$

Define the two constants  $\gamma \equiv (T_1 - T_2)/T_2$  which is positive, and  $r \equiv (C_1 + C_2)/C_1$  which must be larger than unity. Divide both sides of equation (4) by  $C_1$ . Also divide the numerators and denominators under the logarithms by  $C_1 T_2$ . Then substitute in  $T_1/T_2 = 1 + \gamma$  and  $C_2/C_1 = r - 1$  to transform equation (4) into

$$\left(1 + \frac{\gamma}{r}\right)^r \stackrel{?}{>} 1 + \gamma. \quad (5)$$

If  $\gamma < r$  then the left-hand side of equation (5) can be expanded in a binomial series as

$$\left(1 + \frac{\gamma}{r}\right)^r = 1 + r \cdot \frac{\gamma}{r} + \frac{r(r-1)}{2} \left(\frac{\gamma}{r}\right)^2 + \frac{r(r-1)(r-2)}{3!} \left(\frac{\gamma}{r}\right)^3 + \dots \quad (6)$$

so that equation (5) requires demonstrating that

$$\frac{r(r-1)}{2} \left(\frac{\gamma}{r}\right)^2 + \frac{r(r-1)(r-2)}{3!} \left(\frac{\gamma}{r}\right)^3 + \frac{r(r-1)(r-2)(r-3)}{4!} \left(\frac{\gamma}{r}\right)^4 + \dots \stackrel{?}{>} 0. \quad (7)$$

The first term in this series (which has a finite number of terms only when  $r$  is an integer) is positive because  $r > 1$ . But if  $2 > r > 1$  then the second term is negative and the third term is positive, resulting in an alternating series with terms of decreasing magnitude (because both the binomial coefficients and the powers of  $\gamma/r$  separately decrease in magnitude). On the other hand, if  $3 > r > 2$  then the first two terms in equation (7) are positive, and the series thereafter alternates in sign. The idea is the same for larger values of  $r$ . In all cases, the series is convergent with a positive sum [3], which completes the verification.

This method needs to be modified if  $F \equiv \gamma/r > 1$  so that equation (5) becomes

$$(1 + F)^r \stackrel{?}{>} 1 + rF. \quad (8)$$

First, note that

$$(1 + F)^r = F^r \left(1 + \frac{1}{F}\right)^r > F^r \left(1 + \frac{r}{F}\right) = F^r + rF^{r-1}, \quad (9)$$

where the middle inequality follows from the case of equation (5) that was already verified above. Second, observe that

$$F^{r-1} > 1 \quad (10)$$

because both  $F$  and  $r$  are larger than 1. As a result,

$$\begin{aligned} F^{r-1} - r > 1 - r &\Rightarrow F(F^{r-1} - r) > 1 - r \Rightarrow F^r - 1 > r(F - 1) \\ &\Rightarrow F^r - 1 > r(F - F^{r-1}) \Rightarrow F^r + rF^{r-1} > 1 + rF \end{aligned} \quad (11)$$

and substituting the final inequality into equation (9) verifies (8).

Proving equation (5) for the remaining possibility that  $\gamma = r > 1$  (which requires showing that  $2^x > 1 + x$  for any real  $x > 1$ ) can be done graphically [4] by simultaneously plotting  $2^x$  and  $1 + x$  versus  $x$ . It has now been established that the entropy change in equation (3) is positive for the irreversible heat transfer between the blocks. For the special case that  $C_1 = C_2 \equiv C/2$  where  $C$  is the total heat capacity of the system, equation (3) reduces to

$$\Delta S = C \ln \frac{T_{\text{ari}}}{T_{\text{geo}}}. \quad (12)$$

Here  $T_{\text{ari}} \equiv (T_1 + T_2)/2$  is the *arithmetic* mean of the two initial temperatures, whereas  $T_{\text{geo}} \equiv \sqrt{T_1 T_2}$  is their *geometric* mean. According to equation (2), the equilibrium temperature of the system in this case is equal to  $T_{\text{ari}}$ . It can be shown that  $T_{\text{ari}} > T_{\text{geo}}$  by squaring both sides, again establishing that the entropy increases during the thermalization. For example, if  $T_1 = 100$  K and  $T_2 = 900$  K, then  $T_{\text{ari}} = 500$  K  $>$   $T_{\text{geo}} = 300$  K.

It is worth emphasizing to students that unlike energy and momentum, which are strictly conserved for an isolated system, entropy can increase. However, entropy of an isolated system cannot decrease. Conservation of a quantity means it neither increases nor decreases in value, so that entropy *partly* obeys these conditions, in the sense that it complies with one but not the other restriction. Consequently entropy can be said to be ‘paraconserved’ to use a term coined by John Denker [5].

## References

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